Solution 11

1. Use Baire category theorem to show that transcendental numbers are dense in the set of real numbers.

   Solution. A number is called algebraic if it is a root of some polynomial with integer coefficients and it is transcendental otherwise. Let $\mathcal{A}$ be all algebraic numbers and $\mathcal{T}$ be all transcendental numbers. We know that $\mathcal{A}$ is a countable set $\{a_j\}$. Let $\mathcal{A}_n = \{a_1, \cdots, a_n\}$ so $\mathcal{A} = \bigcup_n \mathcal{A}_n$ is a countable union of closed and nowhere dense sets $\mathcal{A}_n$. Hence $\mathcal{A}$ is of first category. As $\mathcal{T}$ is the complement of $\mathcal{A}$, it is a residual set. Since $\mathcal{R}$ is complete, $\mathcal{T}$ is dense by Baire category theorem.

   Alternatively, you may argue that the complement of each $\mathcal{A}_n$ is open and dense, and since $\mathcal{T}$ is the intersection of all these complements, by Baire category theorem, any countable intersection of open dense sets in a complete metric space is dense, hence $\mathcal{T}$ is dense.

2. A set E in a metric space is called a perfect set if, for each point $x \in E$ and $r > 0$, the ball $B_r(x) \bigcap E$ contains a point different from x.

   (a) For each x in the perfect set E, there exists a sequence in E consisting of infinitely many distinct points converging to x.

   (b) Every complete perfect set is uncountable. Hint: Use Baire Category Theorem.

   (c) Is (b) true without completeness?

   Solution. (a). For each $n \geq 1$, as $(B_{1/n}(x) \setminus \{x\}) \bigcap E$ is nonempty, we pick a point from it to form $\{x_n\}$. Obviously, there are infinitely many distinct points in this sequence and it converges to x as $n \to \infty$.

   (b). Assume on the contrary that the perfect set E is countable, $E = \{a_n\}, n \geq 1$. We have $E = \bigcup_{n=1}^{\infty} \{a_n\}$. Obviously every $\{a_n\}$ is a closed set. On the other hand, every ball containing a_n must contain some points different from a_n. We conclude that every $\{a_n\}$ is a closed set with empty interior. However, by assumption, (E, d) is a complete metric space. By Baire Category Theorem E cannot have such decomposition. Therefore, it must be uncountable.

   Note. Applying to $\mathbb{R}$, it gives another proof that $\mathbb{R}$ is uncountable.

   (c). No. Simply consider $\mathbb{Q}$ under the Euclidean metric. It is a countable perfect set which is not complete. Think of the Cauchy sequence $\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \cdots\}$ which is in $\mathbb{Q}$ but converges to π.

3. Optional. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

   (a) Show that $\|x\| \leq C\|x\|_2$ for some C where $\|\cdot\|_2$ is the Euclidean metric.

   (b) Deduce from (a) that the function $x \mapsto \|x\|$ is continuous with respect to the Euclidean metric.

   (c) Show that the inequality $\|x\|_2 \leq C'\|x\|$ for some C' also holds. Hint: Observe that $x \mapsto \|x\|$ is positive on the unit sphere $\{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ which is compact.

   (d) Establish the theorem asserting any two norms in a finite dimensional vector space are equivalent.

Solution. (a). Let $x = a_1e_1 + \cdots + a_ne_n$. By Cauchy-Schwarz Inequality

$$\|x\| = \|\sum_k a_k e_k\| \le \sum_k |a_k| \ \|e\|_k \le C\|x\|_2 \ ,$$

where

$$C = \sqrt{\sum_k \|e_k\|^2} \ .$$

(b). Let $x_n \to x$ in $\|\cdot\|_2$, that is, $\|x_n - x\|_2 \to 0$. By (a), $\|x_n - x\| \to 0$ too.

(c). The map $x \mapsto \|x\|$ is continuous and positive on the unit sphere. As the sphere is compact, it has a positive lower bound, that is, $\|x\| \ge \rho > 0$ whenever $\|x\|_2 = 1$. Now, given any non-zero vector x, $x/\|x\|_2$ belong to the unit sphere, so

$$\left\| \frac{x}{\|x\|_2} \right\| \ge \rho \ ,$$

that is, $\|x\| \ge \rho\|x\|_2$.

(d). Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on the finite dim space V. Fix a basis $\{v_1, \cdots, v_n\}$ in V. Every vector x has a unique representation $x = \sum_{k=1}^n a_k v_k$. The map $x \mapsto (a_1, \cdots, a_n)$ is a linear bijection (linear isomorphism) from V to $\mathbb{R}^n$. It induces two norms on $\mathbb{R}^n$ by $\|a\|_a = \|\sum_k a_k v_k\|_a$ and $\|a\|_b = \|\sum_k a_k v_k\|_b$ (using the same notations). From (c) both are equivalent to the Euclidean norm, hence they are also equivalent to each other. Going back to V, we conclude that they are equivalent too.

4. Let P be the vector space consisting of all polynomials. Show that we cannot find a norm on P so that it becomes a Banach space.

   Solution. Let P_n be the vector subspace of P consisting of all polynomials of degree less than or equal to n. Then $P = \bigcup_{n=1}^\infty P_n$. Any norm on P_n is equivalent to the "Euclidean norm": $\|p\| = (\sum_{k=0}^n a_k^2)^{1/2}$ when $p(x) = \sum_{k=0}^n a_k x^k$. Using this fact, one can show that P_n is a closed subspace of P in any norm. On the other hand, it is clear that P_n is nowhere dense. By Baire category theorem, it is impossible to decompose P as a union of nowhere dense sets when its induced metric is complete.

5. Let $\mathcal{F}$ be a subset of $C(X)$ where X is a complete metric space. Suppose that for each $x \in X$, there exists a constant M depending on x such that $|f(x)| \le M, \ \forall f \in \mathcal{F}$. Prove that there exists an open set G in X and a constant C such that $\sup_{x\in G} |f(x)| \le C$ for all $f \in \mathcal{F}$. Suggestion: Consider the decomposition of X into the sets $X_n = \{x \in X : |f(x)| \le n, \ \ \forall f \in \mathcal{F}\}$.

   Solution. By assumption, $X = \bigcup_n X_n$. It is clear that each X_n is closed. By the completeness of X we appeal to Baire Category Theorem to conclude that there is some n_1 such that X_{n_1} has non-empty interior, call it G. Then $|f(x)| \le n_1, \ \forall x \in G$, for all $f \in \mathcal{F}$.

6. Optional. A function is called non-monotonic if if is not monotonic on every subinterval. Show that all non-monotonic functions form a dense set in $C[a, b]$. Hint: Consider the sets

$$\mathcal{E}_n = \{f \in C[a, b] : \exists x \in [a, b] \text{ such that } (f(y) - f(x))(y - x) \ge 0, \ \forall y, \ |y - x| \le 1/n\}.$$

   (Extend f to $\mathbb{R}$ by setting $f(x) = f(a), x < a, = f(b), x > b$.)

Solution. We will show that each $\mathcal{E}_n$ is closed and . Let $f_k \to f$ uniformly and x_k satisfy $(f_k(y) - f_k(x_k))(y - x_k) \geq 0$ for $y \in [x_k - 1/n, x_k + 1/n]$. By passing to a subsequence, one may assume $x_k \to x_0$. Then

$$\begin{aligned} 0 &\leq (f_k(y) - f_k(x_k))(y - x_k) \\ &= (f_k(y) - f(y))(y - x_k) + (f(y) - f(x_k))(y - x_k) + (f(x_k) - f_k(x_k))(y - x_k) \ . \end{aligned}$$

Letting $k \to \infty$, we obtain $0 \leq (f(y) - f(x_0))(y - x_0)$, hence $\mathcal{E}_n$ is closed.

Next, pick a polynomial p satisfying $\|p - f\|_\infty < \varepsilon/2$. We claim that there exists some g, $\|p - g\|_\infty \leq \varepsilon/2$, does not belong to $\mathcal{E}_n$. Then $\|f - g\|_\infty < \|f - p\|_\infty + \|p - g\|_\infty < \varepsilon$, which shows that $\mathcal{E}_n$ is nowhere dense. Let φ be the jig-saw function that is described in our notes such that $\varphi([a, b]) = [-1, 1]$ and slope equal to a large number $\pm K$ and consider $g = p + \varepsilon/2\varphi$. For $x \in [a, b]$, we can find some $y, |y - x| < 1/n$, such that $(\varphi(y) - \varphi(x))(y - x) \leq -K/3(y - x)^2$.

$$(g(y) - g(x))(y - x) = (p(y) - p(x) + \frac{\varepsilon}{2}(\varphi(y) - \varphi(x))(y - x) \leq L(y - x)^2 - \frac{\varepsilon}{2}\frac{K}{3}(y - x)^2.$$

(L is a Lipschitz constant for p.) By choosing K such that $L - \varepsilon K/6 < 0$, we get $(g(y) - g(x))(y - x) < 0$. In other words, g does not belong to $\mathcal{E}_n$.

We have shown that $\mathcal{E}_n$ is closed and nowhere dense. Similarly, we can show that the set $\mathcal{F}_n = \{f \in C[a, b] : \exists x \in [a, b] \text{ such that } (f(y) - f(x))(y - x) \leq 0, \ \forall y, \ |y - x| \leq 1/n\}$ is closed and nowhere dense. If f does not belong to $\mathcal{E}_n \cup \mathcal{F}_n$, in every interval of the form $[x - 1/n, x + 1/n], x \in [a, b]$, there are points y_1, y_2 such that $(f(y_1) - f(x))(y_1 - x) < 0$, and $(f(y_2) - f(x))(y_2 - x) > 0$. No matter what the relative positions of y_1, y_2 are, one verifies that f is not monotone on $[x - 1/n, x + 1/n]$. Now, if f does not belong to $\mathcal{E}_n \cup \mathcal{F}_n$ for all n, f is not monotone on every interval of the form $[x - 1/n, x + 1/n], x \in [a, b], n \geq 1$. Since every subinterval of $[a, b]$ must contain an interval of the form $[x - 1/n, x + 1/n]$, these f are non-monotonic. According to Baire Category theorem, these functions are dense in $C[a, b]$.